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1998 J. Phys. A: Math. Gen. 31 9855

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# Exact solutions to the Schrödinger equation for the potential $V(r) = ar^2 + br^{-4} + cr^{-6}$ in two dimensions

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Received 20 August 1998

**Abstract.** Making use of an *ansatz* for the eigenfunctions, we obtain an exact closed-form solution to the non-relativistic Schrödinger equation with the anharmonic potential,  $V(r) = ar^2 + br^{-4} + cr^{-6}$  in two dimensions, where the parameters of the potential,  $a$ ,  $b$  and  $c$  satisfy some constraints.

## 1. Introduction

The exact solutions to the fundamental dynamical equations play crucial roles in physics. It is well known that exact solutions to the Schrödinger equation are possible only for several potentials and that some approximation methods are frequently applied to arrive at the solutions. On the other hand, in recent years, the higher-order anharmonic potentials have attracted much more attention from physicists and mathematicians [1–3]. Interest in these anharmonic oscillator-like interactions stems from the fact that, in many cases, the study of the relevant Schrödinger equation, for example in atomic and molecular physics, provides us with insight into the physical problem in question.

Recall that in three-dimensional spaces, roughly speaking, there are two main methods to be used to deal with the anharmonic potentials  $V(r) = ar^2 + br^{-4} + cr^{-6}$ . One [4, 5] is based on an *ansatz* for the eigenfunctions to obtain an exact solution with this potential. This method undoubtedly provides an exact solution for the ground state but sometimes with some constraints on the parameters of the potential. The other method [6, 7] relies on a Laurent series *ansatz* for the eigenfunctions, which converts the Schrödinger equation into a difference equation and then the continued fraction solutions are defined. This method, however, does not give any constraints for the parameters of the potential.

The reason we write this paper is as follows. On the one hand, with the advent of growth technique for the realization of the semiconductor quantum wells, the quantum mechanics of low-dimensional systems has become a major research field: almost all of the computational technique developed for three-dimensional problems has already been extended to lower dimensions. On the other hand, study of the potential  $V(r) = ar^2 + br^{-4} + cr^{-6}$  in two dimensions has never appeared in the literature. We now attempt to research it in two dimensions.

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This paper is organized as follows. In section 2, we study the ground state of the Schrödinger equation with this potential using an *ansatz* for the eigenfunctions. The first excited state will be discussed in the same way in section 3. Some constraints on the parameters of the potential  $a, b, c$  are given in sections 2 and 3. The figures for the unnormalized radial functions are plotted in the last section.

## 2. The ground states

Throughout this paper the natural unit  $\hbar = 1$  and  $\mu = \frac{1}{2}$  are employed. Consider the Schrödinger equation with a potential  $V(r)$  that depends only on the distance  $r$  from the origin

$$H\psi = \left( \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right) \psi + V(r)\psi = E\psi \quad (1)$$

where the potential

$$V(r) = ar^2 + br^{-4} + cr^{-6} \quad a > 0, \quad c > 0. \quad (2)$$

Owing to the symmetry of the potential, let

$$\psi(r, \varphi) = r^{-1/2} R_m(r) e^{\pm im\varphi} \quad m = 0, 1, 2, \dots \quad (3)$$

where the radial wavefunction  $R_m(r)$  satisfies the radial equation

$$\frac{d^2 R_m(r)}{dr^2} + \left[ E - V(r) - \frac{m^2 - \frac{1}{4}}{r^2} \right] R_m(r) = 0 \quad (4a)$$

where  $m$  and  $E$  denote the angular momentum and energy, respectively. For the solution of (4a), we make an *ansatz* [4, 5] for the ground state

$$R_{m0}(r) = \exp[p_{m0}(r)] \quad (5)$$

where

$$p_{m0}(r) = \frac{1}{2}\alpha r^2 + \frac{1}{2}\beta r^{-2} + \kappa \ln r. \quad (6a)$$

After calculating, we arrive at the following equation

$$\frac{d^2 R_{m0}(r)}{dr^2} - \left[ \frac{d^2 p_{m0}(r)}{dr^2} + \left( \frac{dp_{m0}(r)}{dr} \right)^2 \right] R_{m0}(r) = 0. \quad (4b)$$

We compare equation (4b) with equation (4a) and obtain the following set of equations

$$\alpha^2 = a, \quad \beta^2 = c \quad (7a)$$

$$\kappa^2 - \kappa - 2\alpha\beta = m^2 - \frac{1}{4} \quad (7b)$$

$$3\beta - 2\beta\kappa = b \quad (7c)$$

$$E = -(2\kappa + 1)\alpha. \quad (7d)$$

It is easy to obtain the values of parameters for  $p_{m0}(r)$  from equations (7a) and (7b) written as

$$\alpha = \pm\sqrt{a}; \quad \beta = \pm\sqrt{c}; \quad \kappa = \frac{1}{2} \pm \sqrt{m^2 + 2\sqrt{ac}}. \quad (8)$$

In order to retain the well-behaved solution at the origin and at infinity, we choose a positive sign in  $\kappa$  and negative signs in  $\alpha$  and  $\beta$ . Accordingly, equation (7c) leads to the following constraint on the parameters of the potential,

$$(b + 2\sqrt{c})^2 - 4c(m^2 + 2\sqrt{ac}) = 0. \quad (9)$$

The eigenvalue  $E$ , however, will be given from equation (7d) as

$$E = \sqrt{a} \left( 4 + \frac{b}{\sqrt{c}} \right). \quad (10)$$

Now, the corresponding eigenfunctions equation (5) can be read as

$$R_{m0}(r) = N_0 r^\kappa \exp[-\frac{1}{2}(\sqrt{a}r^2 + \sqrt{c}r^{-2})] \quad (11)$$

where  $N_0$  is the normalized constant. Here and hereafter  $\kappa = \frac{1}{2} + \sqrt{m^2 + 2\sqrt{ac}}$ .

### 3. The first excited states

In the same spirit, we make an *ansatz* for the eigenfunctions corresponding to the first excited state in the potential (2) as follows

$$R_{m1}(r) = f_m(r) \exp[p_{m1}(r)] \quad (12)$$

with  $f_m(r)$  given by

$$f_m(r) = a_1 + a_2 r^2 + a_3 r^{-2} \quad (13)$$

and  $p_{m1}(r)$  given by

$$p_{m1}(r) = \frac{1}{2}\alpha_1 r^2 + \frac{1}{2}\beta_1 r^{-2} + \kappa_1 \ln r. \quad (6b)$$

Briefly, it is easy to see from equations (12) and (13) that the radial function  $R_{m1}(r)$  has the following relation

$$R_{m1}(r)'' - \left[ p_{m1}(r)'' + (p_{m1}(r)')^2 + \left( \frac{f_m(r)'' + 2p_{m1}(r)' f_m(r)'}{f_m(r)} \right) \right] R_{m1}(r) = 0 \quad (4c)$$

where the prime denotes the derivative of the radial function with respect to the variable  $r$ . Calculating equation (4c) carefully and comparing it with equation (4a), we obtain

$$a_2[E - \sqrt{a}(2\kappa_1 + 5)] = 0, \quad a_3[b - \sqrt{c}(2\kappa_1 - 7)] = 0 \quad (14a)$$

$$a_1[E - \sqrt{a}(2\kappa_1 + 1)] = a_2 \left[ m^2 - \frac{1}{4} + 2\sqrt{ac} - \kappa_1^2 - 3\kappa_1 - 2 \right] \quad (14b)$$

$$a_1 \left[ m^2 - \frac{1}{4} + 2\sqrt{ac} - \kappa_1^2 + \kappa_1 \right] = a_2[b - \sqrt{c}(2\kappa_1 + 1)] + a_3[-E + \sqrt{a}(2\kappa_1 - 3)] \quad (14c)$$

$$a_1[b - \sqrt{c}(2\kappa_1 - 3)] = -a_3 \left[ m^2 - \frac{1}{4} + 2\sqrt{ac} - \kappa_1^2 + 5\kappa_1 - 6 \right] \quad (14d)$$

$$\alpha_1 = \pm\sqrt{a}, \quad \beta_1 = \pm\sqrt{c}. \quad (14e)$$

Hence, if the angular momentum  $m$  of the first excited state is the same as that of the ground state, we obtain from equation (14)

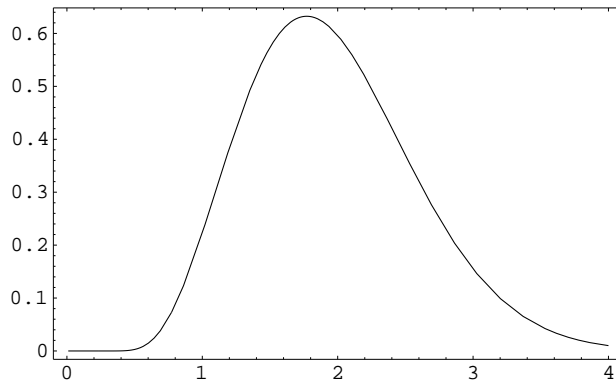
$$\kappa_1 = \frac{b + 7\sqrt{c}}{2\sqrt{c}} \quad E_1 = \sqrt{a}(5 + 2\kappa_1) \quad (15a)$$

$$a_1 = 0 \quad a_2 = \sqrt{a} \quad a_3 = -\sqrt{c} \quad (15b)$$

$$\beta_1 = -\sqrt{c} \quad \alpha_1 = -\sqrt{a} \quad (15c)$$

$$b = -6\sqrt{c} \quad (15d)$$

where the constants both  $\alpha_1$  and  $\beta_1$  are given negative signs in order to maintain the well-behaved nature of the solution at  $r \rightarrow 0$  and  $r \rightarrow \infty$ . Equation (15d) is another constraint on the parameters of the potential.



**Figure 1.** The ground state wavefunctions  $R_0^{(0)}(r)$  as a function of  $r$  for the potential (2) with the values  $a = 1.0$ ,  $b = -12$  and  $c = 4$  in two dimensions. The y-axis denotes the values of radial wavefunctions and the x-axis denotes the variable  $r$ .

At last, the eigenvalue  $E_1$  and eigenfunctions  $R_{m1}(r)$  for the first excited state with the potential (2) may be read from equations (15a) and (12) as follows

$$E_1 = \sqrt{a} \left( 12 + \frac{b}{\sqrt{c}} \right) \quad (16)$$

$$R_{m1}(r) = N_1 (a_2 r^2 + a_3 r^{-2}) r^{\kappa_1} \exp\left[-\frac{1}{2}(\sqrt{a}r^2 + \sqrt{c}r^{-2})\right] \quad (17)$$

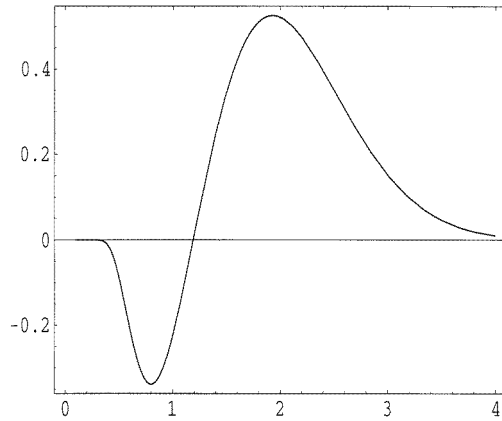
where  $N_1$  is the normalized constant for the first excited state and  $\kappa_1$  is given by equation (15a).

As a matter of fact, the normalized constants  $N_0$  and  $N_1$  can be calculated in principle from the normalized relation

$$\int_0^\infty |R_{mi}|^2 dr = 1 \quad i = 0, 1. \quad (18)$$

Considering the values of the parameters of the potential, we fix them as follows. The value of parameter  $a$  is first fixed: for example  $a = 1.0$ . The values of the parameters  $c$  and  $b$  are determined by the constraints equation (9) and equation (15d) for  $m = 0$ . In this way, the corresponding parameters turn out to be  $a = 1.0$ ,  $c = 4$ ,  $b = -12$ ,  $\kappa = 2.5$ ,  $\kappa_1 = 0.5$ ,  $a_2 = 1$ ,  $a_3 = -2$ . The ground state and the first excited state energies corresponding to these values are obtained as  $E_0 = -2$  and  $E_1 = 6$ , respectively. Actually, when we study the properties of the ground state and the first excited state, as we know, the unnormalized radial wavefunctions will not affect the main features of the wavefunctions. We have plotted the unnormalized radial wavefunctions  $R_0^i$ , ( $i = 0, 1$ ) in figures 1 and 2 for the ground state and the first excited states, respectively. Comparing them with the figures for the ground state and the first excited state in three dimensions, respectively, it is easy to find that they are different from each other. The reason is that the parameters of the potential,  $b$  and  $c$ , are not same as those in three dimensions. This arises from the different constraints on the parameters of the potential, even if the parameter  $a$  is the same in both two and three dimensions.

To summarize, we discuss the ground state and the first excited state for the Schrödinger equation with the potential  $V(r) = ar^2 + br^{-4} + cr^{-6}$  using a simpler *ansatz* for the eigenfunctions; simultaneously two constraints for the parameters of the potential are arrived at from the compared equations, which then results in the variety for the energy eigenvalue and eigenfunctions with the varieties of the parameters of the potential. This simple and



**Figure 2.** The ground state wavefunctions  $R_0^{(1)}(r)$  as a function of  $r$  for the potential (2) with the same parameter values in two dimensions. The  $y$ -axis denotes the values of wavefunctions and the  $x$ -axis denotes the variable  $r$ .

intuitive method is easily generalized. Other studies, for the sextic potential and the octic potential as well as the inverse potential in two dimensions are in progress.

### Acknowledgments

This work was supported by the National Natural Science Foundation of China and grant no LWTZ-1298 from the Chinese Academy of Sciences.

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